Mathematics Teacher

DEVOTED TO THE INTERESTS OF MATHEMATICS IN JUNIOR AND SENIOR HIGH SCHOOLS

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THE MATHEMATICS TEACHER

VOLUME XXI JANUARY, 1928 NUMBER 1

INTRODUCTION TO THE INFINITE

BY DAVID EUGENE SMITH, LL.D.

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Our Attitude to the Infinite.—It is not a matter of much moment that we should or should not teach the various methods of factoring an expression in the form of $ax^2 + bx + c$. Most of the common methods are worthless when we come to the actual use of these factors in the theory of algebraic polynomials, and any application of the work to a "real" problem in algebra is rarely seen. What is said of this detail of elementary algebra may be said of various others; they have a value, but in many cases this value is relatively slight; each concerns a single unimportant muscle of the body algebraic, but it scarcely touches the soul.

Elementary mathematics should never become mere drudgery, a science in which the student has irksome tasks which lead to nothing worth his while; it should always stand forth as a science in which we may imagine a soul to exist; a science which should touch and elevate the souls of all who come in contact with any of its branches.

In elementary mathematics we loosely speak of infinity as a number without limit, or as a line which has no end, and we tell our pupils that it is represented by a symbol, ∞. We thus give a definition without meaning to those who hear it, and a sign without significance to those who see it.

Our duty should be something far different. Just so far as we can adapt our work to the minds of our pupils we should make the effort to get much nearer the soul of mathematics than is the case in most of our classes under present conditions.

What Large Numbers Mean.—As a preliminary to the consideration of the Infinite, it is desirable that we refer briefly to

the meaning of certain large numbers, attaining to some realization of this meaning by the familiar device of comparison.

Professor Moulton, of the University of Chicago, publicly stated not long since that the average life of a solar system, like ours has been calculated to be about a million billion years, and that the age of our planet is about two billion years. Basing his work upon the rate of radium disintegration, Dr. Boltwood had previously computed the age of the earth as slightly less, —say 1,600,000,000 years; and upon these figures Dr. MacCurdy has estimated the length of time since life appeared, in paleozoic time, as 60,000,000 years. These men rank high as scientists, but whether their estimates are correct or not they may serve our purposes as reasonable approximations. Taking them as such, if we represent the entire life epoch on the earth as one revolution of the minute hand of a clock, then a minute-space on the dial will represent 1,000,000 years. In this case, the 6,000 years since the dawn of history in Egypt will be represented by less than ½ second, an arc too short to be seen in the motion of the minute hand. No wonder that the Psalmist said of old, "What is man, that Thou art mindful of him?" and again, "For a thousand years in Thy sight are but as yesterday when it is past, and as a watch in the night."

Consider also what great distances mean. From the sun to Saturn is 886,000,000 miles, nearly ten times as great as the distance to the earth. This seems a vast space for light to traverse; but Neptune is 2,750,000,000 miles away, a distance so great that it takes its light 4 hr. 6 min. to reach us; a distance so great that our minds entirely fail to grasp it except as we resort to this measurement of the time required by light to traverse its enormous reach.

This is but a step, however, when we come to measure the distance to even the nearest of the fixed stars. Between 1831 and 1839 the distance to α Centauri was measured and it was found that it takes the light of this star 4.5 years to reach us; the ordinary methods of recording length fail to have any significance, but roughly speaking we may say that 4.5 years is $4.5 \times 365 \times 6$ times as long as it takes the light of Neptune to reach us, or nearly 10,000 times as long. This, however, is one of the nearest of the fixed stars. Since these measurements were taken, 350 others among the millions of such stars in our galaxy

have been carefully studied and the approximate distances of about 200 are known. The astronomers now call a star relatively near if its distance is within 100 light years—that is, if the light from these stars reaches us with in a hundred years! Spectroscopic methods of measurement give the distances of certain double stars up to 1,000 light years. We now know that our galaxy is only one of an unknown and probably unknowable number of similar universes in the great cosmos. Each of these is composed apparently of hundreds of millions of suns like our own, the nearest of these universes being a million light years away. Such recent discoveries and calculations give reason for credence in some of the earlier speculations of 10,000,000 light years as the distance to the remotest stars that earth dwellers can ever hope to see. In any case, however, the distances which are now accepted as reliable give us information of space far beyond our powers of comprehension. When we think of a light day as 180 times the distance to the sun, and a light year as 65,700 times that distance, and recognize the 1,000 light years represent 65,700,000 times 93,000,000 miles, our imagination revolts at any effort to realize the meaning.

If, in our efforts to realize the unrealizable, we resort to physics, we are equally lost. We can imagine a rifle bullet as leaving the muzzle at the rate of about a mile in 2 seconds. This means 30 miles a minute, 1800 miles an hour, and 15,768,000 miles a year. But at this tremendous speed it would take such a bullet 380,000,000,000 years to reach the universe next to our own.

If we seek for aid from graphs, our position is no more simple. If we represent the sun as a physical point 0.01 inch in diameter, using a microscope to see it, the distance to the earth is 2 inches; to Neptune, 5 feet; to α Centauri, over 2 miles; to the relatively near fixed stars, 50 miles; and to certain double stars whose distance is approximately known, 500 miles. If the distance to the remotest stars that earth-dwellers can hope to see by present methods be taken as 10,000,000 light years, which some have thought reasonable, then our graph would have to reach 1,500,000 miles, the sun's diameter being 0.01 inch!

There was announced in 1925 the measurement of the diameter of the star Mira as 250,000,000 miles, its distance from us being 165 light years. If the center of this star were at the center of the sun, its surface would be 33,000,000 miles beyond our

orbit; that is, our planet Earth would have a diameter only 1/31,000 of that of Mira, and Earth would be hardly more than a pebble lying 33,000,000 miles below the surface.

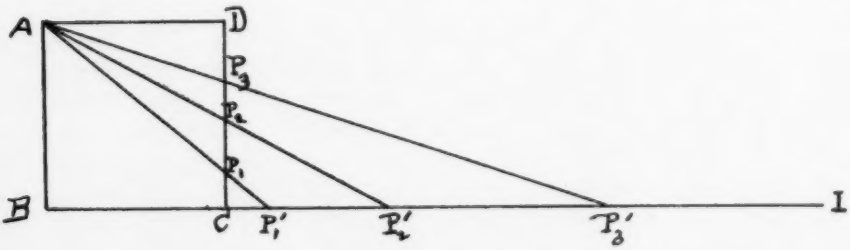
If we should imagine an airplane starting from Earth and traveling continuously at the rate of 100 miles an hour, and able to traverse space beyond our atmosphere, it would take it 27,000,000 years to reach the nearest fixed star, and 1,000,000,000 years to reach the star Mira, which is comparatively near to us.

All this, however, is finite; most of the distances are approximately measurable; in comparison with the infinite they are negligible. Tennyson has expressed the mathematical principle in his lines:

> "Thou hast not reached a real height,
> Nor art thou nearer to the light,
> Because the scale is infinite."

Lucretius, the Roman philosopher-poet of two thousand years ago, expressed the same idea when he wrote:

> "Nor by prolonging life do we take one tittle from the time passed in death, nor can we fret anything away whereby we may haply be a less long time in the condition of the dead."



An Illustration. Let us, however, pass from these thoughts of the finite to thoughts of the infinite, and see to what we are led. If a square *ABCD* stands upon a ray, or half-line *BI* as here shown, and if we join *A* to any point P_1 in *CD* and produce AP_1 to meet *BI* as at P_1', we may say that P_1' corresponds to P_1. Stated otherwise, there is one point and only one in *BI* that thus corresponds to P_1; and conversely, there is one point and only one in *CD* that thus corresponds to P_1'. Similarly, P_2' corresponds to P_2 and vice versa; P_3' corresponds to P_3 and vice versa and so on for all points in *CD* and for all points in *CI*.

In other words, there is a one-to-one correspondence between the points in *CD* and those in *CI*.

But this being the case, there are as many points in *CD* as there are in *CI*, for if we select any point in *CI*, say P_n', and draw AP_n', this line will determine a point in *CD* that is different from any point determined by *A* and any other point P_m' on *CI*. However long *CI* may be, whether a foot, a mile a million miles, or longer, for any point whatsoever in *CI* there is one and only one point in *CD* that corresponds to it, and this point corresponds to no other point in *CI*.

We may now lay *CD* upon *CI* and the same relation must hold true, namely, that for every point on *CD* there is another and a corresponding point on *CI*, and vice versa, and hence the number of points in *CD* is the same as the number of points in the ray *CI*.

At first thought our mind revolts against this statement. We say that *CI* is infinitely longer than *CD* and hence must have infinitely more points, forgetting that a point has no dimensions, and that between any two points, however slight the distance separating them, we can imagine as many points as we please. The fact that we have so often represented a point by a thin solid of ink or chalk has subconsciously led us to think that a point occupies space, and so our senses deceive us.

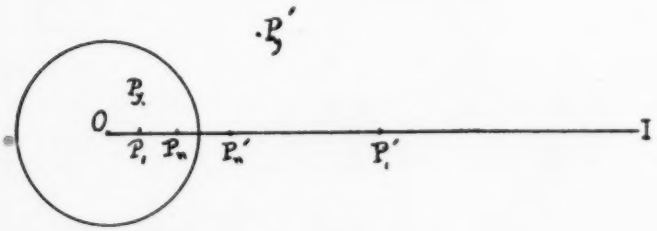
The proof, however, is absolute; we cannot avoid its conclusions; we may rebel, but to no purpose; our little, finite minds have come in touch with the infinite and our little, finite ideas must yield to the truth which the Infinite thrusts upon us.

What is the Infinite?—When we attempt to define the infinite we are confronted by a familiar phenomenon, namely, that the definition, while simple enough in form, does not tend at first sight to elucidate the idea. Professor C. J. Keyser, to whose essays we are all indebted for their dignity of statement, profundity of thought, and lucidity of style, has stated the definition in these words: If we "consider the sequence (*S*), and with it the similar sequence (*S'*),

(*S*) 1, 2, 3, 4, 5, 6, 7,···
(*S'*) 2, 4, 6, 8, 10, 12, 14,···

we see that all the terms of (S') are in (S) and that (S) contains terms that are not in (S'). (S') is, then, a proper *part* of (S). Next, observe that we can pair each term in (S) with the term below it in (S'). That is to say: the whole (S), is equivalent to one of its parts, (S'). A class that thus has a part to which it is equivalent is said to be infinite, and the number of things in such a class is called an infinite number."

The axiom, then, that a whole is greater than any of its parts is valid only for finite magnitudes; when we deal with infinites, the whole may be equal to one of its parts. But the line segment *CD*, although finite has features resembling



those of the infinite ray *CI*; in particular, it has as many points as *CI*, this number being infinite. Thus there comes to be new meaning that the Infinite "make man in His own image," and there comes to be new significance in the idea that every human may have something of the divine within himself.

A Further Illustration.—There are other simple methods of establishing a one-to-one correspondence between the points in finite space and those in infinite space, and one of these is not without interest in elementary teaching. If we lay off a unit segment *OA* on the ray *OI*, and take OP, equal to $\frac{1}{3}$, then there is a point P_1' such that $OP_1' = 3$, P_1 and P_1' being so related that $OP_1 \times OP_1' = 1$. Similarly, for any other point P_n there is Point P_n', and only one, such that $OP_n \times OP_n' = 1$. In other words, there is this one-to-one correspondence between the points on *OA* and those on the ray *OI*. If we take a point P_x' on *OI* equal to 1,000,000, there is a point P_x on *OA* equal to 1/1,000,000 that corresponds to it, and vice versa. As P_x' moves ever so little on on *OI*, P_x moves in unison with it on *OA*.

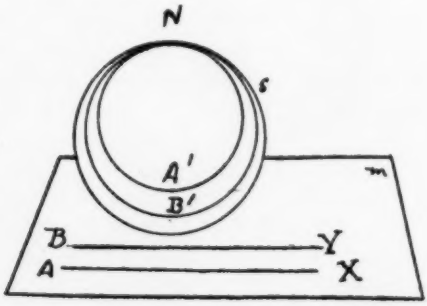
If, now, we revolve OI about O, we shall form a circle, and for any point P_y inside the circle there is a single Point P_y' outside the circle which corresponds to it; and for any point P_y' outside the circle there is a single point P_y inside the circle that corresponds to it. There are, then, as many points inside the circle as there are in the plane outside the circle. Again our minds may show their finite nature by rebelling; but a little thought will lead the infinite that is within us to assert itself and we shall see that there is no escape from the conclusion thus rigorously reached.

Further, if we revolve the plane about OI we shall form a sphere of center O and radius OA. Then to any point P_1 inside the sphere there is one point P_1', outside the sphere, and only one point, that corresponds to P_1. Thus there are as many points within the sphere, however small we take the unit OA, as there are in all the universe about it. Again our finite minds rebel at the idea, but the infinite asserts itself, prejudice gives way to the desire for truth, and we are forced to conclude that the argument is mathematically sound and that finite ideas change when confronted by the infinite.

Picturing the Infinite.—There are many ways of picturing the infinite, some of which have already been mentioned. For example, if we take the radius of the above sphere as unity, and take P_1 at a distance of $\frac{1}{2}$ from O, then we may locate P_1' in the line of OP, by the arbitrary relation $OP_1 \cdot OP_1' = 1$, and hence OP_1' will be equal to 2. The nearer P_1 is to O, the more remote will be P_1', so that as $P_1 \to O$, $P_1' \to \infty$. We have, then, in the point O a point within the circle corresponding to "the point at infinity," and we find ourselves studying the infinite by aid of the infinitesimal.

This leads us to consider for a moment the reciprocals of the enormous distances and bodies in vast space, and to ask ourselves

about minute spaces such as an atom might occupy. Here, too, we are baffled in our search. Within a few years we have come to know that each atom is a minute solar system, with one or more electrons coursing about a central proton. Considered as spheres, we know that an electron is about one five-trillionth of an inch in diameter. Such a number is far too small for our comprehension as is the fact that the number of electrons in the human body is approximately 10^{30}. Physicists tell us that if 2,500,000 persons were to be able to count the electrons passing through a 16-candle-power electric lamp in 1 second, and should divide the work up, each counting 2 electrons per second, it would take 20,000 years to finish.



Abandoning, then, the attempt to comprehend the infinitely small, let us return to the effort to picture the infinitely large. For this purpose we may make use of a simple spherical projection to enable us to obtain some idea of the infinite region. If we let a sphere s rest on a plane m, we may take the north pole N as a center of projection. We may then project on s any figures in m; for example, the parallel lines AX and BY will be projected into $A'N$ and $B'N$. The point N will then correspond to the point at infinity at which we may think of the parallels as meeting, both ways. All such devices have various objectionable characteristics, but for our present need they serve a useful purpose.

If the dweller in the Flatland m says that parallel lines never meet, he is correct in his space; but if his space curves through three dimensions, as in the projection just considered, then the parallel lines meet at infinity, positive infinity is the same as negative infinity, all lines meet in a point at infinity, and we

see with new vision the significance of tan ϕ changing instantly from $+\infty$ to $-\infty$ as ϕ passes through 90°. Similarly, our space of three dimensions may be curved, and the familiar statement may be entirely reasonable,—that a rifle bullet, fired so as to proceed forever, and not deflected from what we imagine to be a straight line, would periodically return to hit us in the back of the head, assuming that we maintained our own position for a time that to us seems infinite.

Significance of this Study.—What is the significance of all this? How does it concern our inner natures? Is it merely a phase of cold mathematics that we shall present in a perfunctory fashion and then assign "the next ten problems for tomorrow"? That depends largely upon ourselves, upon our vision, upon our imagination, upon our conception of the mission of the teacher, and upon our recognition of our opportunities. We may well leave the questions unanswered, for no answer is general; each must answer them for himself. Shall we live our lives to prepare our pupils for examinations upon the square roots of decimals and upon dividing by a polynomial, or shall we open up a vision of some of the things worth while? No one can answer that question for all; but all can answer it for one, —each for himself.

TEACHING BY PARABLES

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Christ was and is the greatest teacher of all time. The people of His day had rejected Him. They had failed to understand His message. They failed to understand Him when He spoke of the essential things for which He was sent among them, so He talked to them of the things with which they were familiar. He endeavored to reveal the Heavenly things through the earthly symbol. He spoke in parables.

In the parable we find similarity of principle but disparity of detail. Understanding is aided by the illustration, both by the likenesses involved and by the comparisons. From our viewpoint it is an attempt to express the new in terms of the old. The purpose is to arrest the attention—to make people listen. The people were not interested in learning that Jesus had come among them to save sinners and that the first step in this process was to get them into the church. They were interested however in His story of the good Samaritan who had compassion on the man who had fallen among thieves and took him to an inn.

We do not need necessarily to speak in finished parables to derive some profit in our teaching from this method used by Christ. When we desire to make, compel, force people to listen, we employ a flag, a brass-band, a picture, a story. Successful advertising compels or entices people into reading the advertisement. The whole subject of interest in the schools is one of bringing the attention to the subject in hand and keeping it there. Christ was attempting to bring the attention of the people to his message and to keep their attention concentrated upon that message.

Many of our fairy stories are parables. They hold the interest of the children by virtue of the story and they convey to the youthful mind much useful information about nature and science and moral conduct. Many of our best novels are of the same nature. Instruction is thus conveyed to a great

mass of people who receive this instruction through their interest in the story. The same people insist on having their medicine made up into sugar coated pills. We are endeavoring these days to convey the same information as in bygone days but we are trying to make the student like the process.

We know that

$$3/10 = {}^{-0.52290}.$$

However we write

$$\log_{10} 3/10 = 9.47710 - 10.$$

That is

$$0.52290 = 9.47710 - 10.$$

Now the student wonders why we insist on writing

$$9.47710 - 10.$$

We can help the student remember what is wanted by saying that we are crazy just like a certain crazy storekeeper who demanded that his bookkeeper make a charge of not less than ten dollars against every man who made a purchase, but being an honest storekeeper asks that his bookkeeper then give credit for an amount such that the books will show the proper amount due the store.

Christ performed many miracles. Whatever His prime purpose in performing these miracles, they helped Him in His teaching. They certainly attracted the attention of multitudes. Some were led to believe in Him through the miracles and believing in Him accepted His teachings. Why should not the present day teacher perform some feat which would obtain for him and his subject the attention, interest, and faith of the student. The teacher of physics or chemistry could well spend a few minutes in performing some pyrotechnical experiment. The mathematician can propose and solve mathematical puzzles—give some short accounts of non-euclidean geometry, of four-dimensional geometry, of the theory of relativity—exhibit as many short cuts as possible. Occasionally employ a method in advance of the student's stage of development, especially if this method enables you to reach a conclusion quicker than the student does by his method. The student will often thus be fired by a desire to know the advanced subject. For example, if he is plotting the curve whose equation is

$$x^2 + y^2 = 25,$$

his method, of solving for y in terms of x, assigning values to x, computing y, plotting the resulting pairs of values of x and y, and joining the points so obtained by as smooth a curve as possible, is a laborious one. You as teacher know at once that the graph is a circle with center at the origin and with radius 5. Mention this fact to the student.

The student of algebra has some difficulty and labor in finding the roots of

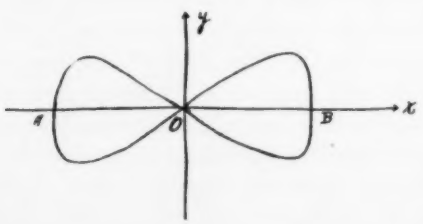
$$x^3 - 1 = 0, \text{ or } x^4 - 1 = 0, \text{ or } x^5 - 1 = 0, \text{ etc.}$$

You, with your knowledge of trigonometry, can take one look at the equation and write down at once the values of the roots and assert that your work is correct.

In the method of parables the proposition in hand is vague. One states an analogous situation with which the student is familiar. The analogy makes the first proposition clear. For example, let it be required to draw the graph of the equation

$$(x^2 + y^2)^2 = a^2(x^2 - y^2).$$

We find that the curve is closed. It is symmetrical with respect to the x-axis, y-axis, and the origin. It crosses the x-axis at $x = a$ and $x = -a$. It crosses the y-axis at the origin only. The tangents at the origin are $y = x$ and $y = -x$. Without further information ask the student to draw in the curve. Some can, others cannot. All can draw the curve however if one states the proposition in the following manner. The x and y axes are to be thought of as streams with bridges at A, O, and B. A jack



rabbit is known to have crossed the bridges at A and B once and only once. He crossed each stream twice at O, always crossing diagonally. For every position that he occupied on one side of either stream, he at some time occupied a symmetrically placed

position on the opposite side. What must have been his path? Any good Indian scout could with this explanation draw in the curve as indicated.

Let us propose the solution of the equation

$$x^2 + 6x = 16$$

We direct the student to complete the square. We find

$$x^2 + 6x + 9 = 16 + 9 = 25.$$

That is, an equation is like unto a scale balance. The equality mark indicates that the $x^2 + 6x$ in one scale pan exactly balances the 16 in the other scale pan. If 9 is placed in the scale pan on the left and the balance is to be maintained one must place 9 (like things) also in the scale pan on the right. Thus one has

$$x^2 + 16x + 9 = 16 + 9.$$

Let there be given three linear equations in three unknowns. Assist the student in remembering the process of solution by making the following statement. A certain man was in the third story of a building and wanted to get to the first story. To get where he wanted to go it became necessary to go first to the second story. So in order to derive from the three equations in three unknowns, one equation in one unknown, one first obtains two equations in two unknowns. Just where the door is situated that leads from each floor to the next lower floor is still an open question.

Let it be required to show that

$$\cos^2 A + 2 \sin^2 A = 1 + \sin^2 A.$$

For $\cos^2 A$ we put its equal $1 - \sin^2 A$ and simplify. We find

$$1 - \sin^2 A + 2 \sin^2 A = 1 + \sin^2 A.$$

This is essentially an economic problem in barter. Suppose one man has some horses and cows and every horse is equal in value to 2 cows. He trades his cows for horses and then on counting his horses finds that he has the same number as his neighbor.

In solving simultaneous quadratic equations some texts enumerate four cases:

1. Both equations are linear in x^2 and y^2.
2. One equation is linear and the other quadratic.

3. Both equations are of the second degree in x and y.
4. Symmetrical equations.

Let there be given the following equations to solve:

$$x^3 - y^3 = 12,$$
$$x - y = 4.$$

These do not come under any one of the four headings above. Divide the first equation by the second. We find

$$x^2 + xy + y^2 = 3.$$

This equation and the equation

$$x - y = 4$$

have the same solutions as the original pair. In general the original problem is replaced by a new problem which we do know how to solve. The unknown has been expressed in terms of the known. The procedure can be likened to that of a Doctor who knows how to cure four diseases. A patient is brought to him who has a different disease. The Doctor proceeds to cure the patient by giving him something which causes the original disease to be replaced by one of the four diseases. Then the Doctor proceeds to cure the patient of this last disease. Thus a cure of the original disease is effected.

My little girl comes running home from school and says "Daddy, please show me how to do my problems." I ask the difficulty. I find that she has been taught how to solve the simple equation

$$n + 2 = 7.$$

The advance assignment is to solve

$$2 + n = 7.$$

To my little daughter this an entirely new problem. What is the difficulty? What shall we say? First, let me say, that the very fact that she thought that this was an entirely new problem was very encouraging. Sometimes, it would seem, she really thinks the commutative law of addition is involved. To her mind it was not clear, and rightly so, that $2 + n$ was equivalent to $n + 2$. The order in many things is material. That there is a very real and important assumption here should be made very

clear. Is the final effect the same when we play in the mud and afterwards wash our hands as it is when we wash our hands and afterwards play in the mud? Every druggist will tell you that the order in which the ingredients are mixed in making an emulsion determines the smoothness of the mixture. Any good cook will tell you that it is not a matter of indifference as to whether in making tomato soup we add the tomatoes to the milk or the milk to the tomatoes. At some stage this commutative law should be made clear. Perhaps in the beginning all that is necessary is to ask the child whether it would make any difference to them whether they had two apples in the left hand and three oranges in the right hand or three oranges in the left hand and two apples in the right hand. This settled the difficulty momentarily for my daughter.

Some students have difficulty in grasping the idea involved in the expressions Highest Common Factor and Lowest Common Multiple. Highest Common Factor can be explained as follows:

One boy has 3 peaches and an orange.

A second boy has 2 peaches and an apple.

What do they have in common? Two peaches.

By lowest common multiple we would mean the smallest number of fruits to be placed on a table such that either boy, but only one, could if he lost his fruit replenish his supply from the table. In this case we would necessarily have on the table three peaches, an orange and an apple.

How shall we correct the student who says that

$$4 \times 0 = 4.$$

I proceed as follows. First, I make the student turn with his back to the board and focus his attention on something else, say my hat. Then, and not until then, I ask the student, if I have no money at all and you have four times as much as I do, how much do you have? To concentrate on any given problem the student must first clear his mind of all other matter. I find that a statement in terms of money is one of the surest ways to make an abstract problem clear.

How shall we insure that the student will remember that division by zero is excluded as a mathematical process. I tell the students that division by zero is like cussing in polite society. It simply isn't done.

The number $a + bi$ is sometimes referred to as an imaginary number. This is a misnomer. This number may represent a kick on the back of your head by the foot of a mule traveling in a given direction with a given force. Such a kick would represent a very real situation. Such a kick would undoubtedly be followed by serious difficulties and might fairly be said to present a *complex* situation.

How shall we impress upon the mind of the student the method of procedure in solving equations containing radicals. For example

$$\sqrt{x} + \sqrt{2x+1} = 5.$$

In general, to solve, it is necessary to isolate the radicals one at a time and get rid of them by squaring. This idea can be made to stick by stating that every time you see a radical you should think of smallpox. The first thing one does with a smallpox patient is to isolate him. The cure comes second. The cure here consists in squaring.

We have

$$e^{\log_e x} = ? \qquad \log_e e^x = ? \qquad (\sqrt{-1})^2 = ? \qquad \sin \arcsin x = ?$$

The question in each instance is the equivalent of, what is the name of the boy whose name is John.

It is sometimes difficult to make clear the idea of orders of infinitesimals. It is sometimes helpful to state that in general differences in the heights of human individuals are comparatively insignificant. People are of the same order of tallness. Races of pygmies exist. These people are not of the same order of tallness as the average American. A man with a million dollars might consider a man with ten thousand dollars as a relatively poor man. Their wealth is not of the same order. A man with ten thousand dollars might consider a man with one hundred dollars as a relatively poor man. In other words, all the fleas have little fleas on their backs to bite them, ad infinitum.

The parable is one of the simplest and clearest methods of catching the attention and making the meaning plain. It is universal in its application. To illustrate the two meanings of Oh. If one drops a brick on your foot you say Oh. If one presents you with a box of candy you again say Oh. But what a difference in meaning.

Quoting from Chapter 13 of the Gospel according to St. Matthew:

"And the disciples came and said unto Him, Why speakest Thou unto them in parables? He an wered and said unto them, because it is given unto you to know the mysteries of the kingdom of heaven, but to them it is not given. . . . Therefore speak I to them in parables: because they seeing, see not; and hearing, they hear not, neither do they understand. . . . All these things spake Jesus unto the multitude in parables; and without a parable Spake He not unto them; that it might be fulfilled which was spoken by the prophet, saying, I will open my mouth in parables; I will utter things which have been kept secret from the foundation of the world."

BY-LAWS OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS (INCORPORATED)

The following preliminary draft of the proposed By-Laws contains a number of legal provisions required in case the Council decides to become incorporated in the State of Illinois. The By-Laws could be considerably condensed if these were eliminated.

All members are asked to give these matters careful consideration and to send all suggestions and criticisms to the Chairman of the Committee.

ARTICLE I—*Name, Purpose, and Corporate Seal*

1. This organization shall be known as
The National Council of Teachers of Mathematics (Incorporated)

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the elementary and secondary fields, by holding meetings for the presentation and discussion of papers, by conducting investigations for the purpose of improving the teaching of mathematics, by the publication of papers, journals, books, and reports, thus to vitalize and coördinate the work of many local organizations of teachers of mathematics and to bring the interests of mathematics to the attention and consideration of the educational world.

3. The corporate seal of the Council shall have inscribed thereon the name of the organization and the words "Corporate Seal—Illinois."

ARTICLE II—*Membership*

1. Any person who is interested in the field of mathematics shall be eligible for election to membership in the Council.

2. Election to membership shall be by a majority vote of the Board of Directors upon written application from the individual seeking admission.

3. Those who were admitted to membership in The N. C. of T. of M. (unincorporated) prior to ....., and were in good standing as such on that date, were thereby admitted to membership in this Council (Incorporated).

ARTICLE III—*Board of Directors and Officers*

1. The Officers of the Council shall be a President, two (2) Vice-Presidents, a Secretary-Treasurer, and the three (3) members of the Committee on Official Journal.

2. The control and management of the affairs and funds of the Council shall be vested in a Board of sixteen (16) Directors (hereinafter called the "Board"), who shall be members of the Council. This Board shall consist of the officers of the Council and nine (9) additional members.

3. The President shall be elected by the Council's members biennially for a term of two years and shall not be eligible for election to a second successive term. The Vice-Presidents shall be elected by the Council's members, one each year for a period of two years and they shall not be eligible for election to a second successive term. (For the first year two Vice-Presidents shall be elected, one for a two-year term one for a one-year term.) Three members of the Board shall be elected by the Council's members annually for a term of three years. (For the first year three members of the Board shall be elected for a three-year term, three for a two-year term and three for a one-year term.) They shall be eligible for reelection but not for more than two consecutive terms. The Secretary-Treasurer and the Committee on Official Journal shall be appointed by the Board for indeterminate terms. For the first year the Committee on Official Journal shall be appointed by the elective members of the Board. All Officers and other Directors shall hold over until their respective successors are elected or appointed and qualify.

4. The Board shall transact the official business of the Council and shall report its actions at the annual business meeting of the Council and in the official journal. A statement regarding any proposed action of the Board which makes or alters a question of policy shall be published in the official journal, or notice of such proposed action shall be mailed to each member, before final action has been taken, so that members of the Council may make known to the Board their individual views. The Board shall appoint from among its members a finance committee of three members who shall have oversight of expenditures under direction of the Board and whose chairman shall countersign all checks drawn by the Secretary-Treasurer.

5. The Board shall have authority to fill vacancies *ad interim*

in any office, including vacancies in the Board and in the Committee on Official Journal, and to make any other appointments necessary for the transaction of the business of the Council.

6. At all meetings of the Board of Directors a quorum shall consist of not less than five (5) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Board, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than announcement at such meeting. Informal action based on a mail ballot by the members of the Board, if ratified at a properly convened meeting of the Board, shall be as valid and effective as if originally authorized at such meeting.

7. At least two months before the date of the annual meeting, all members shall be given the opportunity through announcement in the official journal to suggest by mail for the guidance of the Directors a candidate for each elective office for the ensuing year. At least one month before the annual meeting the Secretary of the Board of Directors shall send to each member an official ballot giving the names of two candidates for each office to be filled. These candidates shall be selected by a nominating committee of the Board of which the Secretary shall be chairman. The election shall be by mail or in person and shall close on the day of the annual meeting.

8. The President shall be the Executive Officer of the Council, shall preside at all meetings of the Board of Directors and at the annual meeting of the Council. He shall have the usual duties pertaining to his office and such other duties as may from time to time be assigned him by the Board of Directors.

9. Either Vice-President shall, in the absence of the President, have and exercise the powers of the President, their order being determined by seniority. The Board of Directors may assign to the Vice-President such duties as may from time to time be determined.

10. The Secretary-Treasurer shall have the usual duties pertaining to the Office of Secretary and of Treasurer, including the custody of the records of the Council and of its Corporate Seal, the keeping of minutes of the meetings of the Board of Directors and of the annual meeting and special meetings of members,

and giving of due notice of all regular and special meetings of the Council and of the Board of Directors and the supervision and safe-keeping of the funds of the Council. The Secretary-Treasurer shall also have the duty of seeing that whenever Directors are elected, including the election of Directors to fill vacancies, a Certificate, under the Seal of the Council, giving the names of those elected and the term of their office, shall be recorded in the Office of the Recorder of Deeds for Cook County, Illinois. Such Certificate shall be signed by the Secretary-Treasurer and verified by oath of the President.

11. The Committee on Official Journal shall have supervision of the official journal subject to the control of the Board of Directors.

Article IV—*Meetings*

1. A meeting of the Council shall be held annually, at such time and place as the Board may direct. Special meetings of the Council may be called from time to time by the President of the Council under authorization of a majority of the Board of Directors to be held at such time and place as may appear from the call.

2. The outgoing Board shall hold a meeting immediately preceding the annual meeting of the Council next succeeding their election, and the members of the new Board shall hold a meeting and organize, by completing the Board, immediately succeeding the annual meeting of the Council at which the new members thereof were elected. Further meetings of the Board may be held from time to time at the call of the President under authorization of five (5) members of the Board.

3. Notice of any meeting of members of the Council shall be given by the Secretary-Treasurer at least thirty (30) days prior to the date set for each meeting. Notice of all meetings of the Board other than the regular meetings provided in Section 2 shall be given to each member of the Board at least fifteen (15) days prior to the date set therefor.

4. Any member of the Council or of the Board may waive notice with the same effect as if due notice had been given him.

5. At all meetings of the Council a quorum shall consist of not less than twenty-five (25) members and no business may be validly transacted at a meeting at which less than a qourum is

present; *provided* that any meeting of the Council, whether or not a quorum be present may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than the announcement at such meeting.

6. Members may take part and vote in person or by proxy at all meetings of the Council.

Article V—*Branches*

1. Any existing association of teachers of mathematics or any newly organized group of not less than fifteen (15) such persons may petition the Board for authority to become a Branch of the Council for the purpose of holding local meetings and of their carrying the work and spirit of the Council to all parts of the country. The Board shall have power to specify the conditions under which such authority shall be granted, and in particular those who constitute a Branch shall at once qualify as members of the Council.

2. The Council shall not be obligated to pay from its treasury any of the expenses of such Branches, but may arrange to do so on such terms as may be agreed upon.

Article VI—*Official Publications*

1. The Council shall publish an official journal, which shall be sent free to all members of the Council in accordance with Article VII.

2. The Board shall have full control of the publication and sale of the official journal and of all other official publications.

3. The official journal shall be under the general management of the Committee on Official Journal but subject to the direction of the Board. There shall also be appointed by the Board a body of Associate Editors who shall give assistance in connection with the official journal and under the direction of the Committee on Official Journal.

4. The Board shall from time to time, as the need arises, make special provision for the management of any other official publications.

5. The Board shall fix the price of the official journal and of any other official publications of the Council, but in no case shall the journal be sold to non-members for less than the annual dues of individual members.

ARTICLE VII—*Dues*

1. The annual dues of each member shall be Two Dollars ($2.00) including a subscription to the official journal. The annual Yearbook and the Register of Members as often as published shall be sold to members at reduced rates as determined from year to year.

2. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list after due notice.

3. New members entering the Council after April 1 of any year shall have their dues prorated for the balance of the year, except when they desire to receive the full current volume of the official journal.

ARTICLE VIII—*Amendments to the Articles of Associations and By-Laws*

1. Changes in the Articles of Association or amendments to the By-Laws may be made at any annual meeting of the Council, or at any adjourned session thereof, or at any special meeting of the Association called for such purpose, by a two-thirds (⅔) vote of those present in person or by proxy and entitled to vote; provided that due notice concerning such amendment shall have been printed in the official journal, or mailed to each member, at least one (1) month before the date of such meeting.

2. No changes in the Articles of Association shall have legal effect until a certificate thereof, verified by oath of the President and under Seal of the Council, attested by the Secretary-Treasurer, shall be filed in the office of the Secretary of State of the State of Illinois and recorded in the office of the Recorder of Deeds for Cook County, Illinois.

Respectfully submitted for preliminary consideration by the members of the National Council of Teachers of Mathematics.

C. M. AUSTIN,
E. W. SCHREIBER,
H. E. SLAUGHT, *Chairman, Chicago University,*
Committee.

ADJUSTING THE COURSE OF STUDY IN NINTH GRADE MATHEMATICS TO THE ABILITY OF THE PUPIL

By HILDEGARDE BECK

Northwestern High School, Detroit, Michigan

This paper is a report of the work of a committee selected last year by Dr. Baker of the Psychological Clinic of the Detroit Board of Education to solve one of the school's most pressing problems, that is, adjusting instructional needs to the slow, average, and fast sections, called "ability" groups that were organized or were to be organized in accordance with the resolutions of the High School principals in our city.

In recent years it has become a truism to say that the school should fit the child rather than that the child should fit the school. We have come at last to admit that there are individual differences in children which make it impossible for them all to progress at the same rate. According to Thorndike's "Psychology of Algebra" the high school pupils of to-day are different from those twenty-five years ago, not only in their experiences and interests, but also in their inborn abilities. He reviews data available for 1918 and compares it with figures for 1890. In 1918 there were six times the number of high school pupils as in 1890. The number graduated, which is in some respects a better measure, was eight times as large in 1918 as in 1890. A fact well worth remembering, Thorndike says, is that almost one in three of the children reaching their teens in the United States enters high school. The corresponding figure for 1890 is not over one in ten.

In referring to the mentality of the pupils of 1890 and 1918, measures were lacking to judge the pupils of 1890. Still he says, "we have, however, excellent reasons for believing that the one in ten had greater capacities for algebra and for intellectual tasks generally than the one in three of to-day."

In our elementary schools provisions have been made for individual differences in children. Dr. Berry,[1] director of Special

[1] *School and Society*, May 9, 1925, "X, Y, Z Pupils in the Detroit Public Schools."

Education in Detroit, in the school year 1919–1920, recommended to the superintendent of schools the adoption of the X, Y and Z groupings of pupils. The recommendation adopted by Superintendent Cody and his administration staff involved the mental classification of all pupils entering the first grade each semester and the gradual extension of such classification through the elementary and the intermediate schools.

The mental classification went steadily forward, grade by grade, until at the present time it is in effect in the B8th grade and by September 1928 will have been carried to the ninth grade. In the Z group we have the lowest 20 percent in intelligence of the pupils. The next 60 percent constitute the Y group; the highest 20 percent the X group. The average I.Q. of the Z group is 83; the Y group is 100; and the X group is 117.

What bearing has the classifying of children into X, Y, and Z groups upon the problem facing us in the 9th grade and throughout high school to-day? It means that more children are entering high school because of fewer failures and more children of low mentality are coming to us than ever before. Should these children of varying ability be put into one group, taught the same work and have the same methods of teaching? The many "misfits" we have, particularly in the 9th and 10th grades, may become adapted partly, if not wholly by ability grouping in the high schools.

Dr. Baker, anticipating the problem facing us, organized various committees in English, History, and Mathematics last year to work on suitable courses of study for the ability groups, for what avail is grouping unless the subject-matter fits the child? The mathematics committee was composed of either heads of departments or some teacher representing the head from each high school respectively. At our first meeting the importance of ability grouping was impressed on us by Assistant Superintendent Edwin L. Miller. Our work was to organize first of all a basic course of study for the Z group. The Y group would continue with the present course of study and that of the X group was to be considerably enriched. To aid us in realizing the psychology of bright pupils and dull pupils so that we would organize work to fit the group properly, Dr. Baker prepared a statement of mental differences which was followed by the suggestions concerning subject-matter that grow out of the

mental differences. It seems like a eulogy of bright pupils and a condemnation of dull pupils but the real purpose was to have us understand the basic characteristics, diagnose them, and then build an educational program around the needs of bright and dull pupils. Dr. Baker's statement is as follows:

"Mental differences have two important aspects with respect to instructional procedures: (1) differences in mental age. (2) differences in brightness or dullness as expressed by I. Q. or in group tests by letter ratings. These two aspects of intelligence present two distinct problems that should be considered separately.

"The mental age of a pupil determines to a large extent the level at which his schooling may be carried. Other factors, such as effort, regular attendance, etc., affect scholarship, but mental age is probably the most important single factor.

"There is a very wide and significant range of mental ages among pupils of the same chronological age. For example, at ten years of age are to be found pupils ranging from seven years to thirteen or fourteen years in mental age. Greater ranges of mental age are to be found among high school pupils but there is a tendency for those of lowest mental age to be eliminated.

"Under the unclassified system of grouping pupils a range of six or eight years of mental ages may be found in practically all small classes or recitation groups. With a classified system this range may easily be reduced to two years or less in all recitation sections.

"Mental age shows the level at which instruction may be carried on. The problem cannot be settled by giving many extra promotions to bright pupils, which merely removes them from their own chronological age group. Instead of this, they need a type of work at their own chronological age level, which is enriched so that it satisfies their mental age and also their chronological and social interests. In like manner, the educational needs of pupils rated D or E at fourteen or fifteen years of age are not adequately met by allowing them to remain associated with normal pre-adolescent pupils of the same mental age in the elementary schools.

"In addition to differences in mental age, the types of learning to be expected of bright or dull pupils are problems of great importance. They have been almost completely neglected in the past.

"For example, two pupils of the same age, one rated A, the other E, not only differ from each other in their mental age but also in the manner of learning. Two pupils of the same mental age, one young and bright, the other old and dull cannot be adequately taught in the same class by the same methods. Bright pupils come to school with high mental ages and a certain type of psychological constitution, dull pupils have low mental ages and a different type of psychological constitution. Some of these differences will be noted in the following statements. While not every dull pupil or every bright pupil shows these characteristics, in the main they represent the tendencies of the group.

"I. Mental dullness or brightness covers all phases of intelligence. That is, dull pupils are not dull in one or two things only, and bright in others, but are dull in all respects. A talent or pseudo-talent occasionally stands out but they are decidedly the exception rather than the rule.

"II. Dull pupils need routine drill in perfecting their learning. Their minds form faulty and inaccurate bonds or associations and the only way to overcome such deficiencies is by a greater amount of repetition and drill. Bright pupils learn by association rather than by rote, so that their learning takes on an entirely different aspect.

"III. Vocabulary differences between bright and dull pupils are very marked. This difference may be due to differences in mental age, but the natural linguistic interests of bright pupils plus a wider range of interests swing the balance definitely in their favor. One of the chief and most pressing needs in the instruction of dull pupils is the simplication of vocabulary in textbooks and in teachers daily use to a point where mastery of this basic tool is actually realized.

"IV. The length of instructional units is another important point upon which bright and dull pupils differ. Short, concise, day to day units are best adapted to dull pupils, whereas long-time projects, with additional references and ulterior goals are more adapted to brighter pupils. This feature alone calls for radical differentiation of the work of bright and of dull pupils.

"V. The physical activity of school projects affects these pupils differently. Dull pupils concentrate their efforts on this aspect at the expense of mental significance, whereas bright pu-

pils react in almost exactly the opposite manner. Unless a bright pupil can get some mental motive out of shop work, of handwriting, etc., he will produce a very mediocre output. Dull pupils enjoy the physical, always at the expense of the mental, although they cannot excel in this direction.

"VI. Dependent versus independent work characterizes the contrast between dull and bright pupils. The dull pupil looks to someone else for his cues and copies those cues in an inadequate and partial manner. Bright pupils wish to plan their own work, wish to find out for themselves, they dislike teacher domination.

"VII. The main issues are comprehended by bright pupils, whereas dull pupils get lost in side issues. The obvious must be pointed out, explained, actually executed and checked in order to insure understanding. Some external and insignificant details catch their attention.

"VIII. Dull pupils usually have a limited and inadequate range of social interests, whereas bright pupils are socially minded. Dull pupils are easily suspicious, selfish, personally offended and hard to be reconciled. Their difficulties on the playground, in the athletic contests, illustrate the point. Bright pupils take a genuine social interest in their environment, and hence get along with people in a much more acceptable manner. The general social status of bright and dull pupils is radically different, but if it could be reversed, bright pupils would still have the advantage.

"IX. In the non-intellectual factors these differences are also evident. Dull pupils are the creatures of instincts and emotions. They prefer these motives to the more difficult intellectual rationalization which bright pupils use. Bright pupils also have instincts and emotions, but they keep them under better control. Bright pupils resent being forced into emotional choices against their better intellectual judgment."

Thus, having the main issues of the problem outlined by Dr. Baker and Mr. Miller, the committee set to work. It voted to meet every two weeks and at these conferences many opinions were exchanged. What should constitute the course of study for the "Z" pupil? Eight major units were decided upon: (1) Graphing; (2) Formulas; (3) Solving of equations; (4) Evaluation of simple algebraic expressions; (5) Finding un-

known distances by scale drawings; (6) Finding unknown distances by similar triangles; (7) Hypotenuse Rule or Law; (8) Positive and Negative Numbers. The course was to include a great deal of drill on the fundamentals of arithmetic.

The committee of sixteen was divided into four small groups and each group was assigned two units. The units were to be rewritten in a simplified form, the vocabulary of which would be within the comprehension of a "Z" mentality. Situations were to be organized or else be of the kind that required little organization. The steps in thinking were to be short and explanations had to be explicit. Facts and questions had to be direct and specific. At the meetings the work on the various units was read, criticized, corrected and then placed in the hands of a teacher experienced in both grade and high school work, who worked the material into a harmonious whole under the direction of Mr. Thiele, Director of Exact Science in the Detroit elementary and intermediate schools.

The course as it now stands is not altogether satisfactory, nor do we claim it will meet the needs of the "Z" pupil altogether. It is in the experimental stage and much more work will have to be done to make the product something of which we can be justly proud.

This preparing of the simplified courses in English, History and Mathematics was but a part of a larger question—how to handle the administrative side of differentiating the curriculum for pupils of different levels of ability. At the end of last spring, Mr. Miller appointed a committee to consider the problems connected with the formation of "slow and fast" classes in high schools. A few of the questions facing this committee are: First, How can grouping be brought about? How many grades of ability are to be recognized by separate groups? Second, How can a program be built that will permit pupils to change from one group to another? Third, What will the standards of promotion be? Would some of the slowest moving sections finally graduate? Will they receive a regular diploma or a new type of diploma? Fourth, What will be the basis on which grouping is done? The factors in selection which we now have are (*a*) Intelligence ratings; (*b*) Scholastic records; (*c*) Chronological age; (*d*) Non-intellectual factors, such as effort, personality and attendance. Then there will be the question of the limits of

ability groups, the size of classes in groups and finally the general organization of the school—the coordination of grade rooms, the programming of classes and the assignment of pupils to classes.

The administrative committee is hard at work at these questions now. The committees in Mathematics, English, and History have prepared the simplified courses, and it will be of much interest to see if we have fitted them to the needs of the pupils of "Z" mentality. Our work is mainly in the making, but already one may note a closer cooperation of all agencies concerned in adjusting the schools to the needs of the pupils.

TWO METHODS OF TEACHING GEOMETRY: SYLLABUS *VS.* TEXTBOOK

By JAMES D. RYAN

Boston Public Latin School

Geometry is intended to be a good course in logical reasoning, and yet, to many, it is a poor course in memory training. Pupils memorize theorems from a textbook (with or without previous explanation by the teacher), recite what they have memorized, but have very great difficulty doing original exercises, even though many of the originals are less difficult than the theorems which have been so glibly recited. Dissatisfied with this situation, many teachers have kept textbooks out of the hands of their pupils, and have taught each theorem as an original. Some issue a printed syllabus of the semester's work; others assign work from day to day, giving the class very little information as to the general direction in which they are going. The general intention in both cases is the same: the pupil must work out each theorem, without at any time seeing it worked in full in a textbook. Absence of a textbook is the important feature of the syllabus method.

The teacher who intends to teach geometry from a textbook finds the opening pages full of definitions of such terms as "line, straight line, curved line, broken line, etc."—terms three fourths of which are thoroughly familiar to the pupil, but which do not in any way arouse his interest in the new subject of geometry. These are followed by a list of axioms, which contain in rather formal language ideas that he is perfectly willing to accept, having already had most of them in algebra.

If, on the other hand, instead of discussing definitions, or dictating a list of axioms, the teacher gives the pupils a straightedge and a pair of compasses, directs him to draw a line on his paper and then to find its mid-point without using any of the markings on the ruler, he has given that pupil a challenge, and that challenge will be accepted. The pupil quickly learns

this construction, and is then asked to describe what he has done. His description will, of course, be couched in non-mathematical language, so that the teacher will have an opportunity to introduce and give correct definitions for such terms as "arc," "intersect," "bisect," etc. Here at the very beginning then, the fundamental constructions can be taught, and a mathematical vocabulary gradually built up. With these fundamental constructions as a background, simple constructions involving lines, angles, triangles, and parallelograms can be done at home. Any new terms which arise are defined by teacher and class working together, and the agreed form written in a blank book.

After the class has done enough construction work to get an idea of the language and tools of Mathematics, the teacher can start on the work leading up to the first theorem: "If 2 straight lines intersect, the vertical angles are equal." As that phase of development would undoubtedly be the same, whether the teacher used textbook or syllabus, I shall not dwell on that now. What I do wish to emphasize is that in the proof of this first theorem, the pupil has to supply *reasons*, and in searching for these reasons, he is introduced to the axioms. In a textbook the axioms are grouped in what the author considers a *logical* arrangement. For example, the axiom "the whole is greater than any of its parts" is usually learned early in the course, yet it is not quoted as a reason until the theorems on inequalities are studied. When the opportunity to use the axiom does arrive, the pupil has probably forgotten it, and has to have it recalled to his mind.

If, on the other hand, the pupil has no text, he learns his axioms as a theorem calls for them. The first two axioms to appear on the list in the pupil's blank book are usually: "Things equal to the same thing are equal to each other" and "If equals are subtracted from equals, the remainders are equal," for these two appear in the first theorem. As the need for other axioms appears in succeeding theorems, they are discussed and written in the blank book. By the time the pupil has studied the first theorem, he has in his notebook such definitions, axioms, and postulates as he has had occasions to use, and no others. In addition, he has in another section most of the fundamental constructions, each on a separate page, with the construction described in accurate language, but with the lower half of the page left blank for the proof which is to come later.

In a class using a text, the teacher usually explains parts of the difficult theorems in advance. Then the pupils take the books home and study the proof, sometimes being obliged to write it out in full, in order to show that they have studied it. In the older texts the proofs were printed in full. The pupils read the words of the author, accepted his reasoning and suggestions, and arrived at the end with no clue as to the author's method, and with no idea how to attack a similar proof. Their greatest mental effort came in memorizing it. Even in the newer texts, where parts of the proof are left to the student, most of the thinking is done by the author, and the student merely "follows the leader." Furthermore, while the teacher is developing the next day's theorem, if a pupil knows that he will find the whole proof in the text, he doesn't worry if some point in the teacher's explanation escapes him; he can find the whole proof in the book.

Contrast the textbook type of assignment (even the best type) with the following: "Construct a quadrilateral $ABCD$ with $AB = CD$ and $AB \parallel CD$. What sort of figure is it? Write a theorem covering this situation and prove it." This calls for ability to construct, and to write a generalized theorem, and in addition, to write a theorem which has never been explained. The theorem is thus made an original exercise and is worked out independently of textbook or teacher. Most difficult theorems are first introduced in class, developed by teacher and class working together, and then completed by the pupils at home. Naturally, a development lesson under these circumstances lends itself to the Socratic method. Very often an original is assigned, worked at home, put on the board, and is then revealed to the class as the next theorem or corollary.

After a proof of a given theorem has been discussed and corrected, the pupils are held responsible for its reproduction, just as if it were written out in a textbook. Proofs are more easily remembered when first learned without a text, but the pupils must have some place where they can refer to them for review. The correct proofs may be copied in a blank book from a model on the blackboard, or copied in a blank book from the teacher's dictation, or mimeographed by the teacher and inserted in pupil's loose-leaf notebook. The first two methods I used the first year that I taught without a text. The chief ob-

jections to them are two: (1) much valuable class time is taken up by copying; (2) neat, accurate copying is hard to obtain. The past three years I have been mimeographing the proofs and thus saving much class time, which can be devoted to work on originals. I have a typewriter with mathematical keys instead of some of the business characters, and with it I cut stencils, and mimeograph the forms the next day in school. The mimeographed forms are distributed, the geometric figures copied, and the sheets filed in a loose-leaf note book for reference and review. The stencils can be dried off and used again. Thus the pupils have what is in reality a loose-leaf textbook, but they do not get printed proofs until they have first worked out the theorems independently and have seen correct proofs on the board.

To the pupil who learns from the textbook, theorems and originals call for two different types of mental effort. In solving an original, the pupil must first analyze the question, and then draw on his accumulated experience for the principles which will help him solve it. He must look around in his mind and choose the proposition or propositions which fit the case. But, in reading a demonstration in a text, the pupil does not have to look around in his mind and choose among many propositions; he has only to follow one person's line of thinking. Without a textbook the pupil has to treat every theorem as an original exercise. Indeed, he often finds the theorem more difficult than an original. The consequence of working without a textbook is that there no longer exists in the pupil's mind an artificial distinction between doing theorems and doing originals.

The older texts either neglected originals entirely, or made no attempt to tie them up with the proper theorems. The newer texts not only grade their originals very carefully, but also attempt to indicate methods of analysis, and to show pupils what to look for when solving originals. These descriptions and hints are usually found, however, just before the miscellaneous exercises at the end of Book I. Yet the pupil has needed these guides from the beginning. Under the syllabus method, on the other hand, just as soon as a new method of proving two angles equal is revealed by the study of a theorem, this new method is put into the blank book under the proper heading. These headings are illustrated by the following:

I. To prove 2 $\angle$s equal, I must prove they are:
   1. Corresp. sides of $\cong$ $\triangle$s.
   2. Bases $\angle$s of an isosceles $\triangle$.
   3. Etc.

II. To prove 2 lines equal, I must prove they are:
   1. Corr. sides of = $\triangle$s.
   2. Opp. sides of a ||gram.
   3. Etc.

With a fresh addition being made to these lists every few days, the pupil is continually being reminded what to look for, in order to prove an original.

In earlier texts the fundamental constructions were usually found at the end of Book II, probably because they involved the use of the compasses, and their study had to be postponed until after the circle was thoroughly discussed. In the newer texts, the constructions are inserted just as soon as they can be proved: in the case of most of them, immediately after the theorems about congruent triangles. While this is an improvement, it introduces a new topic (constructions) to the pupil just at the time when he is struggling with the early stages of proving theorems. If, on the other hand, the teacher is not hampered by a textbook, he can start the year with construction work and carry it through the whole year. The description of the constructions can be written out in full, as has been described earlier in this paper, but the *proof* can be filled in after the necessary theorems have been reached. Constructions, then, are a part of the daily work from the beginning.

It is not necessary for a teacher using the syllabus method to neglect the textbook entirely. I have a set of geometries in my cabinet, and I put them in the hands of the pupils for a few days at a time for work on originals. I also teach the last half of Book IV and all of Book V by the textbook method. For, by the time the class has reached Book IV in April, the pupils ought to have attained a degree of mathematical maturity which will enable them to analyze theorems, even when they are written out in full. For pupils who are repeating geometry or taking a more advanced course, the textbook method may be more suitable and is certainly more economical of time.

Four years of teaching beginners by the syllabus method has

convinced me of the superiority of this method over the text-book method for classes studying geometry for the first time. The teacher can develop just as much of the advance work as he thinks necessary and can leave the rest to the pupils' independent thought; he can, by skillful questioning, encourage the poorer pupils to participate in the development of the theorems; he can arrange the order of theorems to suit the needs of his class; but most of all, he can remove the artificial barrier which exists in the pupils' minds between the study of theorems and the study of originals. These advantages, to my mind, fully compensate for the time spent on copying or mimeographing proofs.

CIVIC VALUES IN THE STUDY OF MATHEMATICS

By PROFESSOR A. S. ADAMS

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At one time or another, we all have the question from parent or student or both as to just what the average student can expect to get out of the study of mathematics. As a brief, and necessarily incomplete, answer to this question, the following paper is offered. My thought has been to present facts and ideas which, while seeming to only touch the vital idealistic views of the subject which would appeal to the educator, will nevertheless mean something to the parent and student.

It seems that Professor Inglis' statement of the aims of secondary education as enunciated in his "Principles of Secondary Education" are generally accepted as a broad and logical analysis. An attempt is made, therefore, to show that the study of mathematics directly satisfies these aims, which are:

1. The preparation of the individual as a prospective citizen and coöperating member of society; *i.e.*, the social-civic aim.
2. The preparation of the individual as a prospective worker and producer; *i.e.*, the economic-vocational Aim.
3. The preparation of the individual for those activities which, while primarily involving individual action, the utilization of leisure, and the development of personality, are of great importance to society; *i.e.*, the individualistic-avocational aim.

The social-civic aim of education is to turn out a person who is a good citizen in every respect. We have heard the disciplinary value of mathematics shouted from the housetops for so long that the phrase is nearly meaningless. Doesn't it seem possible that our questioner would be more favorably impressed if we told him that the study of mathematics fosters good mental habits? I think that for the most individuals the word discipline has a rather unpleasant connotation. Consequently, then,

a connection is to be shown between good mental habits and good citizenship. It is generally accepted by authorities in this work that the mental habits of neatness, orderliness, accuracy, persistence, and attention result from the successful study of mathematics. The question of mental transfer from mathematics to other subjects is much mooted, thereby showing that the question is not absolute but relative; that is, how much mental transfer exists. It is then easily seen that the mental habit of neatness contributes to personal neatness and civic pride, not forming those habits independently it is true, but acting as a powerful contributory agent. It seems that orderliness might properly be joined with neatness, as their civic manifestations are identical.

The community whose citizens possess an accurate mind is indeed fortunate, for in that community guesses will never be advanced as to the cost of some project for the likelihood of passing a desirable piece of legislation. Only facts which have been determined by experts to be correct could be admitted to public discussions. Again, the faculty of persistence can not but augment the civic good, and in what other subjects do we have the fact brought home so squarely that to achieve we must work. There is no doubt but what the importance of this realization by the citizen can not be underestimated.

Lastly, in the discussion of mental habits, comes what many think to be the most important of all; *i.e.*, attention. The novice soon sees that he can attain to very little mathematical learning unless he gives it his undivided attention. Moreover, the competent instructor will never even let him discover this for himself, but will command his attention on the subject matter from the very start. If it be remembered that in the average community less than 10 percent of the voters dictate the policy, one can not fail to appreciate the evils of inattention in civic progress. If it be remembered that the time-honored and proven maxim "A thing worth doing is worth doing well" depends squarely on the principle of attention, a small part of the value of this mental habit is seen.

In further considering the social-civic values of mathematics, we find that there is a certain intellectual honesty developed. This trait results from the consideration in problems as to that which is essential and that which is irrelevant. You will find the mathematically trained citizen perhaps annoying you by his

persistent "How do you know?", but you must concede that he will not admit into the discussion any fanciful, irrelevant padding. When one realizes how much he must learn to get all the facts, he is not apt to waste time with anything off the subject. In other words, he becomes mentally efficient. The politician who measures his votes by the amount of red fire used at election time is not to blame. His constituents who permit themselves to be so easily led are the ones at fault, for they lack this intellectual honesty which would cause them, intelligent human beings, to laugh at such antics demonstrated as qualification for public office.

Nor is this the only value of intellectual honesty. We are all familiar with the individual who lives beyond his means. He is usually unhappy himself, and is a source of annoyance to those with whom he does business. It seems fair to state that he lacks a sense of values, or, in other words, is trying to deceive himself and the world, but accomplishes neither one. I believe it reasonable to assert that that person is intellectually dishonest. So that we see a value accruing to the individual in the field of domestic economy through the intellectual honesty gained in the study of mathematics.

In 1922 the United States suffered from two great strikes. There is no question that if the striking railroad employees and coal miners of that time understood just what would ultimately happen to them were their demands granted, they would have stayed at work. The interdependence of the economic world is very small when compared with that of the mathematical world. The point is that the function idea is not solely mathematical, it is universal. Just as everything in mathematics depends on some other thing, so is one industry dependent on another in the economic world, and if labor and capital could but realize this interdependence, our present upheavals would be non-existent.

To satisfy the economic-vocational aim of education, mathematics develops the mental habits referred to above, has a definite intrinsic value to business and the professions as shown by investigation, and has a growing value to the social sciences.

The value of the mental habits referred to is rather readily seen in every business, trade or profession, and the special applications will be omitted. The actual intrinsic value to business and the professions is shown by an analysis of the "Report of

the Mathematics Section of the New York Society for the Experimental Study of Education" as made in letter of Apr. 5, 1921, from the society mentioned to Professor J. W. A. Young. The data in this report was collected from the answers to a questionnaire sent to a number of successful business and professional men. Seventy-seven answers were received, of which number 59 percent reported as having been helped by the study of mathematics. Further, 62.8 percent reported as having been helped by the study of geometry. Fifty-eight men reported with one exception that their employees could profit by the additional study of mathematics. This same group of men recommended the study of mathematics for their sons and daughters in about the same ratio as for their employees. The inevitable conclusion to be drawn from such an investigation is that mathematics is distinctly worth while from an utilitarian point of view alone.

The easiest relationship to establish is that between mathematics and the physical sciences. The stimulation of investigation, the erection of a continuous chain of reasoning, the scientific imagination, and the very model of scientific thought are unquestionably immediately derived from the study of mathematics. The part which mathematics has played in the developent of these sciences must necessarily be repeated in the development of the social sciences of psychology, sociology, and economics. For the nature of mathematics is two-fold; as Munn says, "with the one aspect the truths face and have contact with the world of outer realities lying in time and space, while with the other aspect they face and have relations with one another." Thus we find the relationships of the social sciences have a distinctly mathematical nature, and that these expressions have been evolved by the inward aspect of mathematics.

The satisfaction of the individualistic-avocational aim by the study of mathematics is undoubtedly the most vague and inexact, yet it is a phase which should not be neglected or classed as unimportant. The individual whose mental training has given him a sense of balance, or intuitional idea of correctness, as Carson puts it, with respect to art and music is surely on the way toward being, not an artist or musician, but a person capable of appreciating such things. Since the earlier forms of art are clearly dependent on the geometry of nature, and since the

later and more highly developed forms of art have carefully retained the principles of the earlier forms, there is no doubt but what a mathematically trained mind may the more readily grasp the significance of the beautiful things of life, both natural and man-made. It is certain that a person who has acquired the ability to appreciate beauty will spend his leisure more profitably to himself and society than he would otherwise.

To demonstrate that mathematics has a connection with the spiritual life of the individual is indeed difficult, but that some such connection exists can not be doubted. Carson tells us in his "Essays on Education" that, after all, mathematics is founded on a few invulnerable statements that can not be proved at the present stage of learning, for "they are based on the simplest known *description* of our space perceptions." Primitive man was religious because he wished to conciliate and propitiate the Unseen Being who controlled his environment. It may be suggested that modern man, despite the control of his environment, gained directly and indirectly from mathematics, is religious for the same reason. However, the fact remains that all religious propose a few fundamental rules of life based on thousands of years of observation of experience, with the same spatial perceptions peculiar to geometry, or rather, mathematics in general. Therefore a knowledge of the fact that all religious and mathematical life springs from the same fountainhead should illuminate the contemplation of the vastness and permanence of the universe.

To sum up, we may say that the study of mathematics aids the individual to think straight, reason logically, and be honest with himself in his civic, economic and avocational life. He is helped to gain the vital things of life, by learning to analyze and discriminate, and finally reject that which is valueless and wasteful of time and effort. Finally, he becomes acquainted with the eternal rightness of all with which he is surrounded, and so he finds his inspiration to achieve his destiny.

BEGINNING GEOMETRY AND COLLEGE ENTRANCE

By RALPH BEATLEY

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In 1923 the College Entrance Examination Board published its Document 108 embodying a detailed statement of the revised requirements in plane and solid geometry. Those who were charged with the preparation of this document gave heed to a suggestion of the National Committee on Mathematical Requirements [1] with respect to the desirability of introducing the more elementary notions of solid geometry in connection with related ideas of plane geometry, and prepared accordingly the syllabus for Geometry *cd*, the so-called Minor Requirement in Plane and Solid Geometry. Document 108 states that "this requirement is designed to cover the most important parts of plane and solid geometry, in such a way that the preparation for it can be completed in the time usually devoted to the standard requirement in plane geometry," *i.e.*, one year.

It is possible that those who prepared this syllabus did not gauge with sufficient accuracy the amount of material which it was proper to include in a one-year course of demonstrative geometry. If that is so, it should not be difficult to make a suitable adjustment when more evidence is at hand on which to base a judgment. The requirement should probably stand as it is for the time being. Certain schools are interested to give it a fair trial, adequate texts are available, and the colleges are not unsympathetic.

It is common knowledge that solid geometry is less widely taught than formerly. That is, a smaller proportion of students completing plane geometry go on to solid geometry than heretofore. This may simply be the result of the students' preference, freely expressed under elective systems of one sort or another. Even a teacher of mathematics, while not discounting the value of the three dimensional experience, might well question the desirability of devoting a year and a half to demonstrative geom-

[1] See p. 37 of their report published in 1923 under the auspices of the Mathematical Association of America, Inc.

etry in the secondary school. No one knows the relative value of eight months of geometry as compared with five; nor is there any information available as to just how many months the student should demonstrate and argue about geometric abstractions before he attains in reasonable measure to those disciplinary and cultural aims which we like to think our teaching of geometry achieves. There does seem to be widespread agreement, however, that the student should not bring his geometric experience to a close without having considered the three dimensional aspects of the world about him. Even before the recent decline in the popularity of solid geometry, our instruction in the mathematics of three dimensions—in colleges as well as in secondary schools—suffered in comparison with that of European schools. We do not need to remain forever complacent about this unfavorable comparison, simply because it is now so familiar to us.

So long, however, as the requirements of the College Entrance Examination Board permit of a choice between Plane Geometry (*C*) and Plane and Solid Geometry, Minor Requirement (*cd*), and so long as human nature remains human, teachers of geometry in secondary schools will tend to continue the same old round and will be slow to follow the lead of those more adventurous spirits who are trying to give in the first—and, for most students, the only—year of geometry not only an adequate training in close reasoning and argumentation, but also an appreciation of the geometry of three dimensions. The number of candidates presenting themselves each year for examination in the minor requirement in plane and solid geometry (*cd*) is pitifully small, and there is good reason to believe that most of these take the examination by mistake. It is not out of place to suggest that the proposal to include a mite of trigonometry, and another mite of logarithms, in the revised requirement of elementary algebra would be today in equally low repute if the new requirement were merely an optional substitute for the old.

Should then this new minor requirement in plane and solid geometry be made the sole requirement, and should the old examination in plane geometry be discontinued? No: that would not be fair. This new requirement in geometry differs from the old far more than was the case in algebra. Furthermore, it is not certain that it is not a bit too long.

Should the new requirement in geometry be withdrawn because so few seek to prepare for it? No: that too would not be

fair. The schools have not had time to give it a fair trial. Even in the case of the trigonometry and logarithms in the algebra requirement, the schools—though meeting the requirements fairly well by now—can hardly be said to have had sufficient time to give a thorough appraisal of the new material and the results of teaching it. No: we need this new requirement constantly before us as an ideal to measure up to. It may possibly prove desirable to modify it very slightly. But to modify it so drastically as to make it tomorrow a fair requirement for all schools, and no longer optional, would cut the heart out of it.

Why not rather modify requirement *C*, making changes gradually over a period of years, and replacing items in plane geometry by items in solid geometry no faster than the traffic can stand? For a beginning, why not expect the candidate to be familiar with simple mensurational work in three dimensions, so that he could meet with confidence a question on the examination paper involving the total area of a house, or cylinder, or silo? Such a question would be necessarily one of the easy originals on the paper. This slight addition to the requirement could be easily balanced by the omission of a few unstarred theorems, or by some other equitable adjustment. In any case this proposal is not intended to make the present requirement easier. The effect of this change would be to shift very slightly the emphasis from "plane geometry only" to "plane and solid geometry." The present ratio of book theorems to originals could—and probably should—stand. At any rate, the emphasis on originals should not be reduced.

This is admittedly only a feeble beginning. But even this little can be very significant. The mensurational work mentioned above has commonly been taught under the head of arithmetic in the eighth grade. It is now appropriately included in the geometry of the junior high school. As such its status is similar to that of the numerical trigonometry now included in the algebra requirement, or of the spelling and grammar implicit in the requirement in English.

Later it ought to be possible to add to the requirement certain book theorems from solid geometry: only a few, if they are to be starred theorems; a dozen or more, if unstarred. In the latter case one of these unstarred theorems could appear in the rôle of an original on the examination paper, just as can now occur in the case of unstarred theorems in plane geometry. How-

ever the change is effected, there should of course be a suitable reduction in the plane geometry part of the requirment.

The experience of the past few years with the requirements as they now stand in Document 108 has indicated the desirability of a few minor changes in the starring of theorems, and the like. When such changes are made it would be a simple matter to incorporate in the present requirement *C* (Plane Geometry) some slight modification in keeping with the suggestion made above. The title of the requirement might then appropriately be changed to "Demonstrative Geometry."

There will always be room for improvements in this and the other syllabi also. As these periodic revisions are made, the change in emphasis in Requirement *C* toward the goal now set for us in Requirement *cd* can be gradually and painlessly accomplished. "In view of the changes taking place at the present time in mathematical courses in secondary schools, and the fact that college entrance requirements should so soon as possible reflect desirable changes and assist in their adoption, the National Committee recommends [2] that either the American Mathematical Society or the Mathematical Association of America (or both) maintain a permanent committee on college entrance requirements in mathematics, such a committee to work in close coöperation with other agencies which are now or may in the future be concerned in a responsible way with the relations between colleges and secondary schools."

At present there is little to encourage the schools to make the change in their teaching of geometry which has been recommended above. It is practically necessary that any such change be accompanied by a simultaneous change in college entrance requirements if it is to be generally effective. This recommendation of a gradual modification of Requirement *C* to approach in character Requirement *cd* has for its main purpose, therefore, the improvement of instruction in geometry in our secondary schools. This is the important thing, and the question of college entrance is raised merely as a means to that end. It would be a shame were we teachers of geometry to witness the gradual disappearance of the extra half year's work in three dimensional geometry from the curriculum without doing anything to save some remnant for the bulk of our students, who seem destined from now on to have but one year to devote to demonstrative geometry.

[2] On p. 48 of its report.

ASSUMPTIONS AND PROOFS

By E. RUSSELL STABLER

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In teaching demonstrative geometry do we aim primarily to develop in our pupils the ability to prove a fixed number of book theorems, or even to prove originals at sight? Assuredly, no. We aim rather to give them a clear understanding of the methods and significance of proofs so that they can apply these methods to non-geometric situations. We cannot hope to accomplish this if we devote our attention solely to geometric theorems and originals. It is evident that some of the work of the class must be related consciously to problems from real life which can be treated to advantage by methods of reasoning which the study of geometry has made familiar.

The following debatable political issue can be made to illustrate the practical utility of "assumption and proof." Ask the class whether or not the United States should join the League of Nations. (I am assuming here that the issue is, for some reason or other, fairly familiar to them.) Suppose those who answer in the negative say, in effect, that we shouldn't enter because it would be contrary to our self-interest, while the affirmative say we should enter because the rest of the world would be helped. Then ask whether the negative is *assuming* that we should act only according to our own interest, and the affirmative that we should be concerned only with helping the rest of the world. They will probably be willing to compromise and say that America should look out both for her own interests and the interests of other countries. This *assumption*—which is agreed upon by all—can be used as a starting point from which the affirmative must prove that we can best help our own interests and those of other countries by going into the League, while the negative must prove that we can help best in each of these ways by staying out.

The principles to be brought out from such a discussion may be listed as follows:

1. When people make statements concerning what policy America should follow, they base those statements consciously or unconsciously on *assumptions* as to how America should behave in relation to other nations.

2. When the assumptions made by different persons are different, the statements or conclusions reached are usually different.

3. For purposes of argument it is valuable that two opposing sides make certain assumptions *in common*. But even then they are unable to reach the same conclusion.

The next step is to ask the class just why we can't be sure of getting just one conclusion when we have all agreed on certain assumptions. Suggest that perhaps those whose families are against the League, or for the League, always manage to reach the same conclusion despite the assumptions. Then—by means of some theorem the truth of which is not obvious without proof —introduce them to geometry as a study where certain assumptions are *always* going to bring the same conclusions regardless of political affiliation or family opinion.

After they have had some experience with geometric methods I should like them to come back to non-mathematical problems of the kind outlined above frequently and in more detail. It would be interesting to try to have debates along strictly geometric lines; at any rate they should get into the habit of arguing geometrically, without bias or prejudice.

There is a danger, of course, in idealizing geometry as drawing certain inevitable conclusions from a set of assumptions. When students are well into the subject they will tend to forget that they ever made any fundamental assumptions and will look upon the theorems as absolute truths. Here is something to be avoided. Suggest to them that we really didn't have to assume the "parallel postulate"—that we might equally well have said that through a given point there are two or no parallels to a given line. In other words, give them an inkling that there is such a science as Non-Euclidean geometry and show them some of the surprising conclusions of this science—such as, that the sum of the angles of a triangle is greater (or less) than two right angles. Bring them to see that the only reason we tend to adopt the Euclidean assumption is that its conclusions seem to conform with our experience better than the Non-Euclidean.

But more important than this is that they should realize the frequent desirability of *changing* assumptions in ordinary life. The cry of the conservative legislator is "it's unconstitutional." Well—what of it? May it not be that we need to revise the assumptions our ancestors made 150 years ago? We are not necessarily bound down in life situations to any fixed list of assumptions yielding (supposedly) absolute truths any more than we are in geometry. The assumptions should be those which are most convenient for our purpose.

We have reason to hope that by methods such as these our pupils can be led to improve the quality of their thinking—not only in geometry, but also in problems from real life quite apart from geometry.

DISCUSSION

The major problem of the teaching of secondary school algebra seems to me to be so important and so insistent just now that we should all of us lend our thought and bend our energy to giving as often as possible a push in the direction of the right solution, and to resisting the very strong impulses in the wrong direction.

This problem arises from the fact that school algebra tends persistently to become too mechanistic, too manipulative; too much a do-so-because-they-say-so sort of subject; too much concerned with the symbols for ideas and not enough with the ideas for which these symbols stand; too much given to the untangling of fictitious assortments of symbols and not enough to the discovering of the purpose of the symbols and their real significance as aids to thinking; too much a routine for mechanical mastery, too little an exercise in discovery and understanding; too much of repetition and not enough of reclarification.

In contemplation this tendency, one of the profound thinkers on the philosophy and teaching of mathematics of this generation, charged school algebra with becoming "a dodge for this and a dodge for that"; and a great philosopher and educator of the past generation said that "algebra is a low form of cunning"; an educator who has achieved much publicity asks repeatedly "Why teach algebra to girls?"; and a professor of education says that algebra is a drill subject like stenography and might well be replaced by stenography as a college entrance requirement; and many parents, pupils, and superintendents of schools adopt a somewhat hostile attitude toward algebra and belittle its educational value.

For this downward tendency on the part of school algebra there are many reasons. It is the easiest path for pupil and teacher and text. It is easy to accept too meagre a view of what our subject is and what it can accomplish. This is often the inherited view, and when we are new teachers we do not always have the inspiring leadership which brings us to see

the real significance of algebra in this modern scientific age. This downward trend is also due to tests which test little but mechanical mastery; which emphasize mere training at the expense of education; it is due to a natural desire for quick and showy results; it is due to the conception of algebra as a myriad of skills (to accept the educational slang of the day) rather than as a symbolism for thought; it is due to the utterly fallacious and thoroughly dangerous notion that the aims and the methods of algebra teaching are comparable to those of arithmetic teaching in grades three to six—a notion which is sometimes accepted but which is based upon a complete misconception of the difference in the modern aims in elementary and in secondary education.

The remedy is difficult, but there is no one move which is doing more to call attention to the need for more thought in algebra and less mechanical substitutes for thought; or to arouse teachers so that they will resist the downward tendency; or to get a constructive counter impulse into the daily practice and thinking of the "average" teacher and the new teacher, than the removal of such words as "cancel" and "transpose" from their vocabulary.

Let those who oppose this carefully planned reform, make sure that they have thought as deeply as the reformers into the major problem of algebra teaching today and into the most effective, practical, and easily administered means of its solution.

HARRY C. BARBER

WHAT ARE THE REAL VALUES OF GEOMETRY?

A Radio Talk

By WINONA PERRY

University of Nebraska

About twenty or even ten years ago, if a person had been asked what parts of geometry he remembered, he might have replied after a few seconds' hesitation that there were many statements which had to be proved, and that for most of them the proofs were given in detail. It is possible that he might have added that he found geometry quite difficult to learn, for memorizing with him was a long and tedious process.

Now, on the contrary, it is likely that anyone who is at all familiar with the present courses in geometry, if asked that same question, as to what parts of geometry he remembered, would answer by asking another question, "Are you referring to intuitive geometry or to demonstrative geometry?" For since the coming of the junior high schools, the boys and girls are being guided in discovering for themselves certain facts concerning the size and shape and position of objects—new ways of measuring lengths and areas and volumes. They learn how to construct and to appreciate designs in art, in clothing, in furniture, in architecture, and in nature. Intuitive geometry is the means whereby a student is able actually to try and see, through experimenting with real objects and situations, what are "the properties of space. . . . It is a laboratory subject. Pupils measure, draw inferences, check up these inferences. They learn to use the methods of science.[1] . . . The typical eleven or twelve year old pupil revels in the explorations of space relations that are opened up to him by the methods of intuitive geometry."[1] An illustration may make this notion of intuitive geometry clearer. A class of boys and girls were asked this problem: "Three house lots 75 feet in width and side by side are owned (one each) by Mr. A, Mr. B, and Mr. C. The other boundary of these lots is a rather steep bank of a stream that crosses these

lots at a given angle. These men wish to have a long fence built along this boundary. How shall the cost of building this fence be divided among these three men?'' The boys and girls measured these boundaries, compared them, and found to their surprise that the same number of feet was contained in each boundary; and, therefore, the cost should be divided equally among these three men. In fact, such ways of learning a fact new to the pupils lead them to become strongly convinced of the truth of this fact. ''The skillful teacher, of course, will gradually lead the pupil to be skeptical of his conclusions, and thereby prepare him for the more certain method of establishing conclusions, the method of demonstration or proof.''[1] In intuitive geometry, then, the students have worked with objects, and have inferred and checked conclusions concerning them and their relations to each other. But although the students have checked these conclusions, they need to know how to prove the correctness of them. Consequently, intuitive geometry is the best possible preparation for the study of the subject more familiar to most adults under the name of demonstrative geometry.

Just what do we mean by demonstrative geometry? In what does it differ from intuitive geometry? In intuitive geometry, the pupil uses the experimental method and discovers and checks the relations between certain facts; but when in addition the student wishes to prove ''beyond the shadow of a doubt'' the general truth of these discovered relations, he has entered the field or province of demonstrative geometry. *Learning by intuition and experience vs. logical proof.* For example, in the illustration mentioned earlier, the students measured the boundary of Mr. A's land, of Mr. B's land and of Mr. C's land. Then they compared the measurements of these boundaries and found that exactly the same number of feet was contained in each boundary. In demonstrative geometry, however, there is found the more powerful and the more satisfying tool; for when the student has proved the relation between those boundaries, he has proved it under generalized conditions. Thence he knows it will still be true, even though the frontage of the lots is less, provided each of the lots has equal frontage; and it will still be true, no matter at what angle the river crosses the lots. In other words, in demonstrative geometry we establish the proofs of statements of facts, and of relations between facts; then each

statement, so proved, may be applicable to many situations which may confront the individual.

Is it possible and is it likely that there will be many occasions in which it will be neeessary for the student to use the information and methods developed in geometry, after the student has left school?

Suppose the student were asked by his father how large to make the hoops for a certain silo which the father is building. This student will know that he can obtain the size desired by multiplying the length of the diameter of the silo, not by three (as has been attempted formerly), but by $3\frac{1}{7}$.

To many instructors and students, however, the actual information obtained in a course in demonstrative geometry is second in importance to the power developed by the individual to draw conclusions and to attack and solve correctly certain types of problem situations. This statement is really a twofold challenge to us: first, will students learn how, through studying demonstrative geometry, to draw and to prove conclusions concerning geometric facts; and secondly, will the student be better able, because of this study of demonstrative geometry, to reason, and to cope successfully with the problematic situations which may confront him outside of school—those in his home, in his community, and in his business life?

With certain statements, the propositions, students find little difficulty usually; but it is quite a different matter in the case of the exercises in geometry (better known as originals)—statements which the students must prove without the guidance of the author. If our first question—*i.e.*, what parts of geometry he remembered—is being answered by an adult, it is not unlikely that he will admit that he was frequently baffled when asked to solve an original; and that when he had succeeded, it was only after long and continuous study. But if he did not reach a successful solution, he had become somewhat discouraged; and to his dismay, he had found his discouragement becoming more frequent; his joy and satisfaction when he attempted to solve originals, less and less. However, at the present time, we have found that it *is* possible to teach students to attack successfully the solution of exercises in geometry. There is no need of haphazard and fumbling attacks, resulting in dissatisfaction because of their previous perplexing failures. "Forewarned is

forearmed," and it *is* possible for teachers not only to be aware of the difficulties which their students will meet, but also to guide their students so that they may be more and more successful in their solutions of exercises in geometry. Furthermore the student in a situation which is baffling him for the time being now can learn how to detect his own errors; and this means that his pile of successes, so to speak, is growing higher and higher. Successful practice is the means whereby students are now becoming more and more able to solve exercises in geometry, and to find increasing satisfaction in their solutions of them.

Almost the more important question is whether the student's ability to draw and to prove conclusions using geometric content will "carry over" in such a way that he will be more able to cope successfully with problems arising in his home and community and business life. Recently it has been shown that students are more able to handle such situations because of their increased power in solving exercises in geometry successfully. This means, for example, that when adults are asked to vote upon certain questions—to do a certain thing or not to do it, to appropriate certain amounts or not to do so, to authorize certain improvements or not to do so, and the like—then those individuals who have developed the abilities in demonstrative geometry through a definite technique in the solution of exercises in geometry will be able to think more intelligently whether it is advisable for them to vote "yes" or "no" on the particular issue before them.

These, then, are the real values to be derived by boys and girls from the study of geometry—information concerning geometric facts, abilities to check the correctness of statements and to draw their own conclusions and verify them in problem situations confronting them.

Bibliography

1. Clark, John R. "Mathematics in the Junior High School," pp. 18–20.
2. Perry, W. M. "A Study in the Psychology of Learning in Geometry."

NEW BOOKS

The Teaching of Junior High School Mathematics. By DAVID EUGENE SMITH and W. D. REEVE. Ginn and Company, 1927. Pp. 411. Price $1.90.

This book is to a considerable extent a pioneer in the field of the teaching of junior high school mathematics, but with none of the evidences of dogmatism or untried theory that frequently accompany a new work of this kind. The wide experience of the authors both as teachers and writers is reflected throughout.

The trend of the book is distinctly favorable to general mathematics, but not in a manner that is likely to offend more conservative teachers. Intuitive geometry, arithmetic, algebra, a unit of demonstrative geometry, and numerical trigonometry are treated under separate chapter headings, with careful regard for the scientific and teaching relations between these topics. The authors have succeeded in observing the spirit of the best committee reports and scientific studies on teaching and have avoided evidences of personal bias or any attempts at propaganda.

The casual reader might occasionally find himself wondering whether the usefulness or the uselessness of mathematics is the main issue, but the book as a whole conveys the idea that was undoubtedly intended; namely, that in elementary mathematics there is plenty of vitally necessary material which the ordinary citizen needs to know, and which is more likely to be learned if the obsolete and the controversial are omitted. The character of the topics discussed, the emphasis upon live subject matter, and the absence of "straw men" indicate that the authors are thoroughly conversant with the advances that have been made in the teaching of mathematics in recent years and with the general and mathematical thought of the times.

Approximately one fifth of the entire book is devoted to objectives. The treatment of this topic is more specific and more exhaustive than that of any other mathematical publication that has appeared to date. Extensive bibliographies appearing at the end of each chapter are chosen with discrimination and contain no apparent omissions of the later literature allied to the main theme of the book. A page or two of topics and questions for discussion are found in each chapter and should prove help-

ful and stimulating to both teacher and pupil. Chapters on model lessons, the place of tests, homemade instruments, mathematical clubs, and recreations are well calculated to arouse interest in a class studying methods, but should prove equally helpful to the individual teacher.

The book is written in a style that is distinctly readable. The topics, in their arrangement and general treatment, are psychologically sound. Even more commendable is the extent to which the point of view of the classroom teacher is represented. The practical attitude of the text and the ease with which it can be followed are calculated to strongly encourage the reading of the book by teachers already in service, while at the same time such qualities are just what are needed in a work that is to be used in classes studying the profession of teaching in schools of education.

J. P. Everett

Teachers College,
Columbia University

Plane Geometry. By William W. Strader and Lawrence D. Rhoads. John C. Winston Company, 1927. Pp. 399.

This book is intended "to provide an interesting and thorough course in plane geometry in accord with the more recent advancements in methods of teaching the subject."

The book is a scholarly bit of workmanship; terms and definitions are unusually consistent and clearly stated. The introductory chapter is carefully organized so as to develop intuitively and inductively the concepts of Geometry. If at first we get the impression that it is a bit too formal, that impression is largely dismissed by a more careful analysis revealing a cleverly laid and simply expressed foundation for the remaining development.

The outstanding features of the text are its unusually excellent drawings and its many interesting pictures showing applications of geometry to life out of school. Add to this its abundance of originals, its supplementary exercises, its chapter summaries and chapter tests, its special labeling of theorems which are recommended by College Entrance Board and the National Committee, and the fact that the book was written by two high school teachers, then you realize that it promises to become a keen competitor with other excellent geometries.

Special attention should be directed to the splendid teaching devices provided at the end of each chapter. Here are drill and test materials, additional exercises to help solve the problem of individual differences, and many interesting and practical applications.

There are a few criticisms which seem evident. The authors have over one hundred theorems and corollaries proved practically in full. This is reactionary and not in accord with recent advancements which suggest a minimum list with many of the former theorems unproved but included as originals, or proofs merely suggested.

The analysis of a proof into statements and "facts" rather than statements and reasons is unfortunate. These "facts" are not all facts, they are merely the rules of the game. Even our axioms and postulates are mere arbitrary statements upon which our system of logic is built, and not necessarily true. There are conditions under which a straight line may not be the shortest distance between two points nor the whole greater than either of its parts.

The authors say on page 3, "Simplicity of proof has been secured . . . by choosing a sequence which has eliminated a number of indirect proofs and any necessity for the commensurable and incommensurable proofs of certain theorems." Purposeful "elimination of many indirect proofs" is of questionable value, because indirect reasoning is one of the most common forms of reasoning used in out-of-school activity. While it would be difficult to defend the contention that a selection of sequence makes any incommensurable case unnecessary in a rigorous geometry, still the omission of these cases is to be highly commended simply because they are too difficult for most high school students to understand.

On the whole the book has unusual excellence, some features indicating conservatism and others a modern progressive attitude. The authors, who are both high school teachers, are to be highly commended on an excellent piece of work whose virtues far outweigh its faults. The book should find wide usage with progressive teachers who are looking for "an interesting and thorough course in Plane Geometry."

H. C. Christofferson

Teachers College,
Columbia University

NEWS NOTES

"Teaching as a Fine Art" will be the keynote of the Eighth Annual Ohio State Educational Conference which the College of Education, Ohio State University, will conduct at Columbus on April 12, 13, and 14. Last year, with an attendance record of 4,500, the Seventh Annual Conference quite shattered all previous registration figures. The Executive Committee for the 1928 Conference anticipates that the "new record a year" tradition will be continued this year.

As in the past, the sectional meetings, held throughout Friday, the second day of the Conference, will feature the three-day session. This provision for group meetings by school people of the State whose interests and problems are similar has met with such evident approval that meetings must now be scheduled, in many instances, in both the morning and afternoon for certain of the sections. In order to avoid the overlapping which necessarily occurs when the sectional program is extended, some of the group meetings have been scheduled this year for Thursday afternoon. This will be the first Conference at which any meetings have been held in advance of the opening general session held on the evening of the first day.

The following sectional meetings have been arranged for the 1928 Conference. Thursday afternoon, April 12: City Superintendents, County Superintendents, School Librarians, and Parent-Teacher Association; Friday, April 13: Art, Biological Science, City Superintendents, Clinical Psychology, Commercial Education, Deans of Women, Educational and Intelligence Tests, Elementary Principals, Elementary Teachers, English, Geography, High-school Principals, History, Home Economics, Industrial and Vocational Education, Journalism, Junior High-school Principals, Kindergarten and Primary Teachers, Latin, Mathematics, Modern Language, Music, Non-Biological Science, Physical Education, Religious Education, School Business Officials, Special Education, Teacher Training, Village and Consolidated School Superintendents.

On the afternoon of October 7 the Mathematics Section of

the Northeastern Wisconsin Teachers Association met at the Oshkosh Normal School. The following program was presented:

I. "Pupil Preparation for Ninth Grade Algebra." Miss Barr of the Oshkosh High School.

II. "The Unit Assignment in Algebra." Mr. Giles of the State Department of Education.

III. "Presenting the Algebraic Graph in Ninth Grade Mathematics." Miss Florence E. Fell, East High School, Green Bay.

Each paper was followed by an animated and helpful discussion.

FLORENCE E. FELL.

ANNOUNCEMENTS

On page 62 there appears a ballot for the election of officers of the National Council of Teachers of Mathematics for the ensuing year. Every subscriber to the MATHEMATICS TEACHER is automatically a member of the Council and is entitled to vote. In fact it is the duty of every member to vote for the officers each year. In no other way can the members be adequately represented.

The results of the election will be given at the annual meeting of the Council to be held at the Statler Hotel in Boston on Friday and Saturday, February 24 and 25. The complete program of the Boston meeting will appear in the February issue of the MATHEMATICS TEACHER.

The general theme of the program is Mathematics and Life and President Gugle has secured some very able speakers. Friday morning will be given over to a meeting of the Executive Committee, Friday afternoon will be devoted to a general business session in which important problems concerning the new constitution and by-laws and incorporation will be discussed.

On Friday evening the regular program of the session begins with an address of welcome and one of response. These will be followed by a keynote address giving somewhat of the historical and philosophical background, showing the part that mathematics has played in the development of various civilizations.

On Saturday morning the program will have two major divisions as follows:

1. Mathematics in Science and Modern Inventions.
    a. In Astronomy.
    b. In Physics and Chemistry.
2. Mathematics and Modern Business.
    a. In Statistics.
    b. In Large Corporations.

The general theme of the Saturday afternoon session is "The Challenge of Modern Life." There are three general divisions as follows:

1. The New Mathematics as a part of the New Education.
   a. Its Nature and Function.
   b. Its Method and Devices.
   c. Its Challenge and Opportunity.
2. Practical application of High School Mathematics.
3. The Presentation of the Third Yearbook entitled "Selected Topics in the Teaching of Mathematics."

A large banquet at the Statler Hotel on Saturday evening with an inspirational address will close what we hope will be the most successful meeting the Council has ever held. Watch for the final program in the February number of the TEACHER and plan to be at the meeting.

THE NATIONAL COUNCIL YEARBOOKS

The attention of our subscribers is called to the yearbooks of the National Council. The First Yearbook on "A General Survey of Progress" may be secured for $1.10 from C. M. Austin, Oak Park High School, Oak Park, Illinois. The Second Yearbook on "Curriculum Problems in Teaching Mathematics" may be secured from the Bureau of Publications, Teachers College, 525 West 120th Street, New York City. The unbound volume is $1.25, the bound $1.50. The Third Yearbook on "Selected Topics in the Teaching of Mathematics" will also be published by the Bureau of Publications, Teachers College, and can be secured on and after February 20th, for $1.75 (Bound Volume). Send in your orders before the supplies are exhausted.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

BALLOT

NOTICE: Mark and mail before February 1, 1928, to the Secretary, Mr. J. A. Foberg, California, Pennsylvania.

FOR PRESIDENT (One to be elected)

- ☐ HARRY BARBER, Boston, Massachusetts
  (Retiring member of Executive Committee)
- ☐ WILLIAM BETZ, Rochester, New York
  (Member of Executive Committee, 1927 to 1930)

FOR VICE-PRESIDENT (One to be elected)

- ☐ C. M. AUSTIN, Oak Park, Illinois
  (Re-election—Vice-President 1 year)
- ☐ ALFRED DAVIS, St. Louis, Missouri

FOR MEMBERS OF EXECUTIVE COMMITTEE
(Two to be elected)

- ☐ HARRY ENGLISH, Washington, D. C.
  (Re-election. Member of committee one term)
- ☐ MARIE GUGLE, Columbus, Ohio
  (Retiring President)
- ☐ EDWIN SCHREIBER, Ann Arbor, Michigan
  (Secretary pro tem at Dallas meeting)
- ☐ ORPHA WORDEN, Detroit, Michigan
  (Former member of Executive Committee)